

# Periodic Solutions of Singular Hamiltonian Systems with Fixed Energies

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## Abstract

We use the variational minimizing method with a suitable constraint and a variant of the famous Benci-Rabinowitz's saddle point Theorem to study the existence of new non-trivial periodic solutions with a prescribed energy for second order Hamiltonian systems with singular potentials  $V \in C^2(R^n \setminus O, R)$  and  $V \in C^1(R^n \setminus O, R)$  which may have an unbounded potential well, our results can be regarded as some complementaries of the well-known Theorems of Benci-Gluck-Ziller-Hayashi and Ambrosetti-Coti Zelati etc..

**Key Words:** Singular Hamiltonian systems, periodic solutions, variational methods.

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## 1. Introduction

For classical second order Hamiltonian systems without singularity, based on the works of Seifert ([33]) in 1948 and Rabinowitz ([29, 30]) in 1978 and 1979, Benci ([8, 9]) and Gluck-Ziller ([19]) and Hayashi ([23]) used Jacobi metric and very complicated geodesic methods and algebraic topology to study the periodic solutions with a fixed energy:

$$\ddot{q} + V'(q) = 0, \quad (1.1)$$

$$\frac{1}{2}|\dot{q}|^2 + V(q) = h. \quad (1.2)$$

They proved a very general theorem:

**Theorem 1.1** Suppose  $V \in C^2(R^n, R)$ , if

$$\{x \in R^n | V(x) \leq h\}$$

is bounded and non-empty, then the (1.1)-(1.2) has a periodic solution with energy  $h$ .

Furthermore, if

$$V'(x) \neq O, \forall x \in \{x \in R^n | V(x) = h\},$$

then the (1.1)-(1.2) has a nonconstant periodic solution with energy  $h$ .

For the existence of multiple periodic solutions for (1.1)-(1.2) with compact energy surfaces, we can refer Groessen([22]) and Long[24] and the references there.

In 1987, Ambrosetti-Coti Zelati[1] used Clark-Ekeland's dual action principle, Ambrosetti-Rabinowitz's mountain pass theorem etc. to study the existence of  $T$ -periodic solutions of the second-order equation

$$-\ddot{x} = \nabla U(x),$$

where

$$U = V \in C^2(\Omega; \mathbf{R})$$

is such that

$$U(x) \rightarrow \infty, x \rightarrow \Gamma = \partial\Omega;$$

where  $\Omega \subset \mathbf{R}^n$  is a bounded and convex domain, they got the following result:

**Theorem 1.2** Suppose that

$$(i). U(O) = 0 = \min U;$$

(ii).  $U(x) \leq \theta(x, \nabla U(x))$  for some  $\theta \in (0, \frac{1}{2})$  and for all  $x$  near  $\Gamma$  (superquadraticity near  $\Gamma$ );

(iii).  $(U''(x)y, y) \geq k|y|^2$  for some  $k > 0$  and for all  $(x, y) \in \Omega \times \mathbf{R}^N$ .

Let  $\omega_N$  be the greatest eigenvalue of  $U''(0)$  and  $T_0 = (2/\omega_N)^{1/2}$ .

Then  $-\ddot{x} = \nabla U(x)$  has for each  $T \in (0, T_0)$  a periodic solution with minimal period  $T$ .

For  $C^r$  systems, a natural interesting problem is if

$$\{x \in R^n | V(x) \leq h\}$$

is unbounded, can we get nonconstant periodic solution for (1.1) – (1.2)?

In 1987, D. Offin[27] firstly generalized Theorem 1.1 to some non-compact cases under  $V \in C^3(R^n, R)$  and complicate geometrical assumptions on potential wells, but the geometrical conditions seem difficult to verify for concrete potentials.

In 1988, Rabinowitz[31] studied multiple periodic solutions for classical Hamiltonian systems with potential  $V \in C^1(R \times R^n, R)$ , where  $V(q_1, \dots, q_n; t)$  is  $T_i$ -periodic in positions  $q_i \in R$  and is  $T$ -periodic in  $t$ .

In 1990, using Clark-Ekeland's dual variational principle and Ambrosetti-Rabinowitz's Mountain Pass Lemma, Coti Zelati-Ekeland-Lions [14] studied Hamiltonian systems in convex potential wells, they got the following result:

**Theorem 1.3** Let  $\Omega$  be a convex open subset of  $R^n$  containing the origin  $O$ . Let  $V \in C^2(\Omega, R)$  be such that

(V1).  $V(q) \geq V(O) = 0, \forall q \in \Omega$ .

(V2).  $\forall q \neq O, V''(q) > 0$ .

(V3).  $\exists \omega > 0, s.t. :$

$$V(q) \leq \frac{\omega}{2} \|q\|^2, \forall \|q\| < \epsilon.$$

$$(V4). V''(q)^{-1} \rightarrow 0, \|q\| \rightarrow 0 \text{ or}$$

$$(V4)'. V''(q)^{-1} \rightarrow 0, q \rightarrow \partial\Omega.$$

Then, for every  $T < \frac{2\pi}{\sqrt{\omega}}$ , (1.1) has a solution with minimal period  $T$ .

In Theorems 1.2 and 1.3, the authors assumed the convex conditions for potentials and potential wells so that they can apply Clark-Ekeland's dual variational principle, we notice that Theorems 1.1-1.3 essentially made the following assumption:

$$V(x) \rightarrow \infty, x \rightarrow \Gamma = \partial\Omega.$$

So that all the potential wells are bounded.

For singular Hamiltonian systems with a fixed energy  $h \in R$ , Ambrosetti-Coti Zelati([3,5]) used Ljusternik-Schnirelmann theory on an  $C^1$  manifold to get the following Theorem:

**Theorem 1.4**(Ambrosetti-Coti Zelati[3]) Suppose  $V \in C^2(R^n \setminus \{O\}, R)$  satisfies:

$$(A1)$$

$$3V'(u) \cdot u + (V''(u)u, u) \neq 0$$

$$(A2)$$

$$V'(u) \cdot u > 0$$

$$(A3) \exists \alpha > 2, \text{ s.t.}$$

$$V'(u) \cdot u \leq -\alpha V(u)$$

$$(A4) \exists \beta > 2, r > 0, \text{ s.t.}$$

$$V'(u) \cdot u \geq -\beta V(u), 0 < |u| < r$$

$$(A5)$$

$$V(u) + \frac{1}{2}V'(u)u \leq 0$$

Then (1.1)-(1.2) has at least one non-constant periodic solution.

After Ambrosetti-Coti Zelati, a lot of mathematicians studied singular Hamiltonian systems, here we only mention a related recent paper of Carminati-Sere-Tanaka[11], they used complex variational and geometrical and topological methods to generalize Pisani's results ([28]), they got

**Theorem 1.5** Suppose  $h > 0, L_0 > 0$  and  $V \in C^\infty(R^n \setminus \{O\}, R)$  satisfies

$$(B1) V(q) \leq 0;$$

$$(B2) V(q) + \frac{1}{2}V'(q)q \leq h, \forall |q| \geq e^{L_0};$$

$$(B3) V(q) + \frac{1}{2}V'(q)q \geq h, \forall |q| \leq e^{-L_0};$$

$$(A4) \exists \beta > 2, r > 0, \text{ s.t.}$$

$$V'(q) \cdot q \geq -\beta V(q), 0 < |q| < r;$$

Then (1.1) – (1.2) has at least one periodic solution with the given energy  $h$  and whose action is at most  $2\pi r_0$  with

$$r_0 = \max\{[2(h - V(q))]^{\frac{1}{2}}; |q| = 1\}.$$

**Theorem 1.6** Suppose  $h > 0, \rho_0 > 0$  and  $V \in C^\infty(R^n \setminus \{O\}, R)$  satisfies (B1), (A4) and

$$(B2') \lim_{|q| \rightarrow +\infty} V'(q) = O.$$

$$(B3') V(q) + \frac{1}{2}V'(q)q \geq h, \forall |q| \leq \rho_0;$$

Then (1.1) – (1.2) has at least one periodic solution with the given energy  $h$  and whose action is at most  $2\pi r_0$ .

Using the variational minimizing method with a constraint ,we get:

**Theorem 1.7** Suppose  $V \in C^2(R^n \setminus \{O\}, R)$  satisfies A1, A2, A3 and (A4)'  $\exists \beta > 2$ , s.t.

$$V'(q) \cdot q \geq -\beta V(q), 0 < |q| < +\infty;$$

(A5)'

$$V(-q) = V(q), \forall q \neq O.$$

Then for any  $h > \frac{\mu_2}{\alpha}$ , (1.1) – (1.2) has at least one non-constant periodic solution with the given energy  $h$ .

Using Benci-Rabinowitz's saddle point Theorem, we get the following Theorem:

**Theorem 1.8** Suppose  $V \in C^1(R^n \setminus \{O\}, R)$  satisfies

(P1)

$$V'(u) \rightarrow O, \|u\| \rightarrow +\infty.$$

(A3)  $\exists \alpha > 2$ , s.t.

$$V'(u) \cdot u \leq -\alpha V(u) + \mu_2.$$

(A4)  $\exists \beta > 2, r > 0$ , s.t.

$$V'(u) \cdot u \geq -\beta V(u), 0 < |u| < r.$$

Then for any  $h > \frac{\mu_2}{\alpha}$ , (1.1) – (1.2) has at least one non-constant periodic solution with the given energy  $h$ .

**Corollary 1.9** Suppose  $\alpha = \beta > 2$  and

$$V(x) = -|x|^{-\alpha}$$

Then for any  $h > 0$ , (1.1) – (1.2) has at least one non-constant periodic solution with the given energy  $h$ .

**Remark** In our Theorem 1.8 ,the assumption on regularity for potential  $V$  is weaker than Theorems 1.1-1.6. Comparing Theorem 1.5 and Theorem 1.6, we don't need (B1), and (A3) is also different from (B2)-(B3) and (B3').

## 2 A Few Lemmas

Let

$$H^1 = W^{1,2}(R/Z, R^n) = \{u : R \rightarrow R^n, u \in L^2, \dot{u} \in L^2, u(t+1) = u(t)\}$$

Then the standard  $H^1$  norm is equivalent to

$$\|u\| = \|u\|_{H^1} = \left( \int_0^1 |\dot{u}|^2 dt \right)^{1/2} + |u(0)|.$$

Let

$$\Lambda = \{u \in H^1 | u(t) \neq O, \forall t\}.$$

**Lemma 2.1**([3]) Let

$$F = \{u \in H^1 | \int_0^1 (V(u) + \frac{1}{2}V'(u)u)dt = h\}.$$

If (A1) holds, then  $F$  is a  $C^1$  manifold with codimension 1 in  $H^1$ .

Let

$$f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 V'(u)u dt$$

and  $\tilde{u} \in F$  be such that  $f'(\tilde{u}) = O$  and  $f(\tilde{u}) > 0$ . Set

$$\frac{1}{T^2} = \frac{\int_0^1 V'(\tilde{u})\tilde{u} dt}{\int_0^1 |\dot{\tilde{u}}|^2 dt}$$

If (A2) holds, then  $\tilde{q}(t) = \tilde{u}(t/T)$  is a non-constant  $T$ -periodic solution for (1.1)-(1.2). Moreover, if (A2) holds, then  $f(u) \geq 0$  on  $F$  and  $f(u) = 0, u \in F$  if and only if  $u$  is constant.

**Lemma 2.2**([3,22]) Let  $f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u))dt$  and  $\tilde{u} \in \Lambda$  be such that  $f'(\tilde{u}) = O$  and  $f(\tilde{u}) > 0$ . Set

$$\frac{1}{T^2} = \frac{\int_0^1 (h - V(\tilde{u}))dt}{\frac{1}{2} \int_0^1 |\dot{\tilde{u}}|^2 dt} \quad (2.1)$$

Then  $\tilde{q}(t) = \tilde{u}(t/T)$  is a non-constant  $T$ -periodic solution for (1.1)-(1.2). Furthermore, if  $V(x) < h, \forall x \neq O$ , then  $f(u) \geq 0$  on  $\Lambda$  and  $f(u) = 0, u \in \Lambda$  if and only if  $u$  is a nonzero constant.

**Lemma 2.3**(Sobolev-Rellich-Kondrachov[26],[41])

$$W^{1,2}(R/Z, R^n) \subset C(R/Z, R^n)$$

and the imbedding is compact.

**Lemma 2.4**([26,41]) Let  $q \in W^{1,2}(R/TZ, R^n)$ .

(1).If  $q(0) = q(T) = 0$ , then we have Friedrics–Poincaré inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

(2).If  $\int_0^T q(t)dt = 0$ , then we have Wirtinger’s inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T}\right)^2 \int_0^T |q(t)|^2 dt$$

and Sobolev’s inequality:

$$\int_0^T |\dot{q}(t)|^2 dt \geq \frac{12}{T} |q(t)|_\infty^2.$$

**Lemma 2.5**(Eberlein-Shmulyan [39]) A Banach space  $X$  is reflexive if and only if any bounded sequence in  $X$  has a weakly convergent subsequence.

**Definition 2.1**(Tonelli [26]) Let  $X$  is a Banach space,  $f : X \rightarrow R$ .

(i).If for any  $\{x_n\} \subset X$  strongly converges to  $x_0$ :  $x_n \rightarrow x_0$ , we have

$$\liminf f(x_n) \geq f(x_0),$$

then we call  $f(x)$  is lower semi-continuous at  $x_0$ .

(ii).If for any  $\{x_n\} \subset X$  weakly converges to  $x_0$ :  $x_n \rightharpoonup x_0$ , we have

$$\liminf f(x_n) \geq f(x_0),$$

then we call  $f(x)$  is weakly lower semi-continuous at  $x_0$ .

Using the famous Ekeland’s variational principle, Ekeland proved

**Lemma 2.6**(Ekeland[16]) Let  $X$  be a Banach space,  $F \subset X$  be a closed (weakly closed) subset. Suppose that  $\Phi$  defined on  $X$  is Gateaux-differentiable and lower semi-continuous (or weakly lower semi-continuous) and assume  $\Phi|_F$  restricted on  $F$  is bounded from below. Then there is a sequence  $x_n \subset F$  such that

$$\Phi(x_n) \rightarrow \inf_F \Phi$$

$$\|\Phi|'_F(x_n)\| \rightarrow 0.$$

**Definition 2.2**([16,18]) Let  $X$  be a Banach space,  $F \subset X$  be a closed subset. Suppose that  $\Phi$  defined on  $X$  is Gateaux-differentiable, if sequence  $\{x_n\} \subset F$  such that

$$\Phi(x_n) \rightarrow c,$$

$$\|\Phi|'_F(x_n)\| \rightarrow 0,$$

then  $\{x_n\}$  has a strongly convergent subsequence.

Then we call  $f$  satisfies  $(PS)_{c,F}$  condition at the level  $c$  for the closed subset  $F \subset X$ , we denote it as  $(PS)_{c,F}$

We can give a weaker condition than  $(PS)_{c,F}$  condition:

**Definition 2.3** Let  $X$  be a Banach space,  $F \subset X$  be a weakly closed subset. Suppose that  $\Phi$  defined on  $X$  is Gateaux-differentiable, if sequence  $\{x_n\} \subset F$  such that

$$\begin{aligned}\Phi(x_n) &\rightarrow c, \\ \|\Phi|_F'(x_n)\| &\rightarrow 0,\end{aligned}$$

then  $\{x_n\}$  has a weakly convergent subsequence.

Then we call  $f$  satisfies  $(WPS)_{c,F}$  condition.

Now by **Lemma 2.6**, it's easy to prove

**Lemma 2.7** Let  $X$  be a Banach space,

Let  $F \subset X$  be a weakly closed subset. Suppose that  $\Phi$  defined on  $F$  is Gateaux-differentiable and weakly lower semi-continuous and bounded from below on  $F$ . If  $\Phi$  satisfies  $(WPS)_{inf\Phi, F}$  condition, then  $\Phi$  attains its infimum on  $F$ .

**Lemma 2.8**(Gordon [20]) Let  $V$  satisfies so called Gordon's Strong Force condition:

There exists a neighborhood  $\mathcal{N}$  of  $O$  and a function  $U \in C^1(\Omega, \mathbb{R})$  such that:

- (i)  $\lim_{s \rightarrow 0} U(x) = -\infty$ ;
- (ii)  $-V(x) \geq |U'(x)|^2$  for every  $x \in \mathcal{N} - \{O\}$ .

Let

$$\partial\Lambda = \{u \in H^1 = W^{1,2}(R/Z, R^n), \exists t_0, u(t_0) = O\}.$$

Then we have

$$\int_0^1 V(u)dt \rightarrow -\infty, \forall u_n \rightharpoonup u \in \partial\Lambda.$$

Let

$$\partial\Lambda = \{u \in H^1 = W^{1,2}(R/Z, R^n), \exists t_0, u(t_0) = 0\}.$$

Then we have

$$\int_0^1 V(u)dt \rightarrow -\infty, \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

By Lemma 2.7 and 2.8, it's easy to prove:

**Lemma 2.9** Let  $X$  be a Banach space,  $F \subset X$  be a weakly closed subset. Suppose that  $\phi(u)$  is defined on an open subset  $\Lambda \subset X$  and is Gateaux-differentiable on  $\Lambda$  and weakly lower semi-continuous and bounded from below on  $\Lambda \cap F$ , if  $\phi$  satisfies  $(WCPS)_{inf\phi}$  condition, and

$$\phi(u_n) \rightarrow +\infty, u_n \rightharpoonup u \in \partial\Lambda,$$

then  $\phi$  attains its infimum on  $\Lambda \cap F$ .

Combining classical Benci-Rabinowitz's Generalized Mountain-Pass Lemma and Gordon's strong force condition and  $(CPS)^+$  condition, we have

**Lemma 2.10**(Benci-Rabinowitz [5], Generalized Mountain-Pass Lemma) Let  $X$  be a Banach space,  $\Lambda$  is an open subset of  $X$ ,  $f \in C(\Lambda, R)$  satisfies  $(CPS)^+$  condition, that is : If  $\{u_n\} \subset \Lambda$  satisfies

$$f(u_n) \rightarrow c > 0, \quad (1 + \|u_n\|)f'(u_n) \rightarrow O. \quad (2.2)$$

Then  $\{u_n\}$  has a strongly convergent subsequence in  $\Lambda$ .

Let  $X = X_1 \oplus X_2$ ,  $\dim X_1 < +\infty$ ,

$$\begin{aligned} B_a &= \{x \in X \mid \|x\| \leq a\}, \\ S &= \partial B_\rho \cap X_2, \rho > 0, \\ \partial Q &= (B_R \cap X_1) \cup (\partial B_R \cap (X_1 \oplus R^+e)), R > \rho, \end{aligned}$$

where  $e \in X_2$ ,  $\|e\| = 1$ ,

$$\begin{aligned} \partial B_R \cap (X_1 \oplus R^+e) &= \{x_1 + se \mid (x_1, s) \in X_1 \times R^+, \|x_1\|^2 + s^2 = R^2\} \\ Q &= \{x_1 + se \mid (x_1, s) \in X_1 \times R^1, s \geq 0, \|x_1\|^2 + s^2 \leq R^2\} \end{aligned}$$

If

$$f|_S \geq \alpha,$$

and

$$f|_{\partial Q} \leq \beta < \alpha,$$

If

$$f(u_n) \rightarrow +\infty, u_n \rightharpoonup u \in \partial\Lambda.$$

Then  $C = \inf_{\phi \in \Gamma} \sup_{x \in Q} f(\phi(x)) \geq \alpha$  and is a critical value for  $f$ .

### 3 The Proof of Theorem 1.7

Let

$$\partial\Lambda_0 = \{u \in H^1 = W^{1,2}(R/Z, R^n), u(t+1/2) = -u(t), \exists t_0, u(t_0) = 0\}.$$

**Lemma 3.1** Assume (A4) holds, then for any weakly convergent sequence  $u_n \rightharpoonup u \in \partial\Lambda_0$ , there holds

$$f(u_n) \rightarrow +\infty$$

**Proof** Similar to the proof of Zhang[40].

**Lemma 3.2**  $F \cap \Lambda$  are weakly closed subset in  $H^1$ .

**Proof** Let  $\{u_n\} \subset F \cap \Lambda$  be a weakly convergent sequence, we use the embedding theorem to know which uniformly converges to  $u \in H^1$ .

Now we claim  $u \in \Lambda$ , and then it's obviously that  $u \in F$ . In fact, if  $u \in \partial\Lambda$ . By condition (A4)' we have



$$-V(u) \geq |u|^{-\beta}$$

So  $V(u)$  satisfies Gordon's Strong Force Condition ,by his Lemma,we have

$$\int_0^1 -V(u_n)dt \rightarrow +\infty, \forall u_n \rightharpoonup u \in \partial\Lambda$$

Condition (A4)' implies

$$V(u_n) + \frac{1}{2} \langle V'(u_n), u_n \rangle \geq (1 - \frac{\beta}{2})V(u_n).$$

Hence

$$h = \int_0^1 [V(u_n) + \frac{1}{2} \langle V'(u_n), u_n \rangle]dt \rightarrow +\infty.$$

This is a contradiction.

**Lemma 3.3**  $f(u)$  is weakly lower semi-continuous on  $F \cap \Lambda_0$

**Proof** For any  $u_n \subset F: u_n \rightharpoonup u$ , then by Sobolev's embedding Theorem, we have the uniformly convergence:

$$|u_n(t) - u(t)|_\infty \rightarrow 0.$$

(i).If  $u \in \Lambda_0$ , then by  $V \in C^1(R^n \setminus \{0\}, R)$ , we have

$$|V(u_n(t)) - V(u(t))|_\infty \rightarrow 0$$

And by the weakly lower semi-continuity for norm ,we have

$$\liminf \|u_n\| \geq \|u\|.$$

Hence

$$\begin{aligned} \liminf f(u_n) &= \liminf \left( \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \right) \int_0^1 (h - V(u_n)) dt \\ &\geq \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u)) dt = f(u). \end{aligned}$$

(ii).If  $u \in \partial\Lambda_0$ , then by  $V$  satisfying Gordon's Strong Force condition, we have

$$\int_0^1 -V(u_n)dt \rightarrow +\infty, \forall u_n \rightharpoonup u \in \partial\Lambda_0.$$

(1).if  $u \equiv 0$  ,then

$$|u_n|_\infty \rightarrow 0, n \rightarrow +\infty.$$

Then similar to the proof in [40], we have

$$f(u_n) \geq 6|u_n|_\infty^{2-\beta} \rightarrow +\infty, n \rightarrow +\infty.$$

So in this case we have

$$\liminf f(u_n) = +\infty \geq f(u).$$

(2).if  $u \neq 0$ . By the weakly lower semi-continuity for norm ,we have

$$\liminf \|u_n\| \geq \|u\| > 0.$$

So by Gordon's Lemma,we have

$$\begin{aligned} \liminf f(u_n) &= \liminf \left( \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \right) \int_0^1 (h - V(u_n)) dt = +\infty \\ &= \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u)) dt = f(u). \end{aligned}$$

**Lemma 3.4** The functional  $f(u)$  has positive lower bound on  $F$

**Proof** By the definitions of  $f(u)$  and  $F$  and the assumption (A2) , we have

$$f(u) = \frac{1}{4} \int_0^1 |\dot{u}|^2 dt \int_0^1 (V'(u)u) dt \geq 0, \forall u \in F.$$

Furthermore,we claims that

$$\inf_{F \cap \Lambda_0} f(u) > 0,$$

since otherwise,  $u_0(t) = \text{const}$  attains the infimum 0, then by the symmetry of  $\Lambda_0$ ,we have  $u_0(t) \equiv 0$ ,which contradicts with the definition and (A4) . Now by Lemmas 3.1-3.4 and Lemma 2.9,we know  $f(u)$  attains the infimum on  $F$ ,and we know that the minimizer is nonconstant .

## 4 The Proof of Theorem 1.8

**Lemma 4.1** Under the assumptions (P1) – (A3) – (A4) of Theorem1.8,  $f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \int_0^1 (h - V(u)) dt$  satisfies  $(CPS)^+$  condition on  $\Lambda$ ,that is, if  $\{u_n\} \subset \Lambda$  satisfies

$$f(u_n) \rightarrow c > 0, \quad (1 + \|u_n\|)f'(u_n) \rightarrow 0. \quad (4.1)$$

Then  $\{u_n\}$  has a strongly convergent subsequence in  $\Lambda$ .

**Proof** Since  $f'(u_n)$  make sense,we know

$$\{u_n\} \subset \Lambda$$

We claim  $\int_0^1 |\dot{u}_n|^2 dt$  is bounded. In fact, by  $f(u_n) \rightarrow c$ , we have

$$-\frac{1}{2} \|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 V(u_n) dt \rightarrow c - \frac{h}{2} \|\dot{u}_n\|_{L^2}^2 \quad (4.2)$$

By (A3) we have

$$\begin{aligned}
\langle f'(u_n), u_n \rangle &= \|\dot{u}_n\|_{L^2}^2 \cdot \int_0^1 (h - V(u_n) - \frac{1}{2} \langle V'(u_n), u_n \rangle) dt \\
&\geq \|\dot{u}_n\|_{L^2}^2 \int_0^1 [h - \frac{\mu_2}{2} - (1 - \frac{\alpha}{2})V(u_n)] dt
\end{aligned} \tag{4.3}$$

By (4.2) and (4.3) we have

$$\begin{aligned}
\langle f'(u_n), u_n \rangle &\geq (h - \frac{\mu_2}{2})\|\dot{u}_n\|_{L^2}^2 + (1 - \frac{\alpha}{2})(2c - h\|\dot{u}_n\|_{L^2}^2) \\
&= (\frac{\alpha}{2}h - \frac{\mu_2}{2})\|\dot{u}_n\|_{L^2}^2 + C_1
\end{aligned} \tag{4.4}$$

Where  $C_1 = 2(1 - \frac{\alpha}{2})c$ ,  $\alpha > 2$ ,  $h > \frac{\mu_2}{\alpha}$ . So  $\|\dot{u}_n\|_2 \leq C_2$ .

Then we claim  $|u_n(0)|$  is bounded.

We notice that

$$\begin{aligned}
f'(u_n) \cdot (u_n - u_n(0)) &= \int_0^1 |\dot{u}_n|^2 dt \int_0^1 (h - V(u_n)) dt \\
&\quad - \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \int_0^1 \langle V'(u_n), u_n - u_n(0) \rangle dt \\
&= 2f(u_n) - \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \int_0^1 \langle V'(u_n), u_n - u_n(0) \rangle dt
\end{aligned} \tag{4.5}$$

If  $|u_n(0)|$  is unbounded, then there is a subsequence, still denoted by  $u_n$  s.t.  $|u_n(0)| \rightarrow +\infty$ . Since

$$\|\dot{u}_n\| \leq M_1,$$

then

$$\min_{0 \leq t \leq 1} |u_n(t)| \geq |u_n(0)| - \|\dot{u}_n\|_2 \rightarrow +\infty, \text{ as } n \rightarrow +\infty \tag{4.6}$$

By Friedrics-Poincare's inequality and condition (P1), we have

$$\int_0^1 |\dot{u}_n(t)|^2 dt \geq \pi^2 \int_0^1 |u_n(t) - u_n(0)|^2 dt, \tag{4.7}$$

$$\int_0^1 V'(u_n)(u_n - u_n(0)) dt \rightarrow 0, \tag{4.8}$$

$$f'(u_n) \cdot (u_n - u_n(0)) \rightarrow 0. \tag{4.9}$$

So  $f(u_n) \rightarrow 0$ , which contradicts  $f(u_n) \rightarrow c > 0$ , hence  $u_n(0)$  is bounded, and  $\|u_n\| = \|\dot{u}_n\|_{L^2} + |u_n(0)|$  is bounded. By Sobolev's embedding inequality, we know it is also

bounded in maximum norm ,by the continuity of  $V$ ,  $V(u_n)$  is also uniformly bounded in maximum norm, so by  $f(u_n) \rightarrow c > 0$  , we have  $d > 0$  such that ,when  $n$  large enough,

$$0 < d \leq f(u_n) = \frac{1}{2} \|\dot{u}_n\|_{L^2}^2 \int_0^1 (h - V(u_n)) dt \leq \frac{e}{2} \|\dot{u}_n\|_{L^2}^2 \quad (4.10)$$

that is

$$\|\dot{u}_n\|_{L^2}^2 \geq \frac{2d}{e} > 0, \quad (4.11)$$

It is easy to know that

$$\langle f'(u_n), u_n \rangle = \|\dot{u}_n\|_{L^2}^2 \int_0^1 [h - V(u_n) - \frac{1}{2} \langle V'(u_n), u_n \rangle] dt \quad (4.12)$$

By (4.1),we have  $\langle f'(u_n), u_n \rangle \rightarrow 0$ ,hence

$$\int_0^1 [h - V(u_n) - \frac{1}{2} \langle V'(u_n), u_n \rangle] dt \rightarrow 0. \quad (4.13)$$

Since  $\|u_n\| = \|\dot{u}_n\|_{L^2} + |u_n(0)|$  is bounded, and we know that  $H^1$  is a reflexive Banach space, so  $\{u_n\}$  has a weakly convergent subsequence,we still denote it as  $u_n$ ,by the embedding theorem, which uniformly converges to  $u \in H^1$ .

Now we claim  $u \in \Lambda$ , in fact,if  $u \in \partial\Lambda$ , there are two cases:

$$(i). u \equiv O.$$

By Sobolev's embedding Theorem,we have

$$|u_n|_\infty \rightarrow 0, n \rightarrow +\infty.$$

Hence condition (A4) implies,when  $n$  is large enough,

$$V(u_n) + \frac{1}{2} \langle V'(u_n), u_n \rangle \geq (1 - \frac{\beta}{2}) V(u_n).$$

By condition (A4) we have

$$-V(u_n) \geq |u_n|^{-\beta}$$

So  $V(u_n)$  satisfies Gordon's Strong Force Condition ,by his Lemma,we have

$$\begin{aligned} \int_0^1 -V(u_n) dt &\rightarrow +\infty, \forall u_n \rightharpoonup u \in \partial\Lambda \\ \int_0^1 [V(u_n) + \frac{1}{2} \langle V'(u_n), u_n \rangle] dt &\rightarrow +\infty. \end{aligned} \quad (4.14)$$

which contradicts with (4.13).

$$(ii). u \neq O.$$

Then  $\int_0^1 |\dot{u}|^2 dt \neq 0$  since there exists  $t_0 : u(t_0) = O$ . By the weakly lower semi-continuity for norm ,we have

$$\liminf [(\int_0^1 |\dot{u}_n|^2 dt)^{\frac{1}{2}} + |u_n(0)|] \geq [(\int_0^1 |\dot{u}|^2 dt)^{\frac{1}{2}} + |u(0)|].$$

By

$$|u_n(0) - u(0)| \rightarrow 0.$$

Then we have

$$\liminf \int_0^1 |\dot{u}_n|^2 dt \geq \int_0^1 |\dot{u}|^2 dt > 0.$$

Since  $V(u)$  satisfies Gordon's Strong Force Condition ,by his Lemma,we have

$$\int_0^1 V(u) dt \rightarrow -\infty, \forall u_n \rightharpoonup u \in \partial\Lambda$$

$$f(u_n) = \frac{1}{2} \int_0^1 |\dot{u}_n|^2 dt \int_0^1 (h - V(u_n)) dt \rightarrow +\infty.$$

which contradicts with  $f(u_n) \rightarrow c < +\infty$ . Hence  $u_n$  weakly converges to  $u \in \Lambda$ . Furthermore, similar to the proof of Ambrosetti-Coti Zelati([5]),  $u_n$  strongly converges to  $u \in \Lambda$ .

**Proof of Theorem 1.8** In Benci-Rabinowitz's Saddle Point Theorem, we take

$$X_1 = R^n, X_2 = \{u \in W^{1,2}(R/Z, R^n), \int_0^1 u dt = 0\}$$

$$S = \left\{ u \in X_2 \mid \left( \int_0^1 |\dot{u}_2|^2 dt \right)^{1/2} = \rho \right\},$$

$$\partial Q = \{u_1 \in R^n \mid |u_1| \leq R\} \cup$$

$$\{u = u_1 + se, u_1 \in R^n, e \in X_2, \|e\| = 1, s > 0, \|u\| = (|u_1(0)|^2 + s^2)^{1/2} = R > \rho\}.$$

By condition (A4) we have

$$-V(u) \geq |u|^{-\beta}$$

So  $V(u)$  satisfies Gordon's Strong Force Condition ,by his Lemma,we have

$$\int_0^1 -V(u_n) dt \rightarrow +\infty, \forall u_n \rightharpoonup u \in \partial\Lambda.$$

By the definition for  $f(u)$  , we have

$$f(u) = \frac{1}{2} \int_0^1 |\dot{u}|^2 dt \cdot \int_0^1 (h - V(u)) dt$$

$$f|_S \geq \frac{1}{2} \|u\|^2 \int_0^1 |u|^{-\beta} dt$$

$$\begin{aligned}
&\geq \frac{1}{2}||u||^2 \int_0^1 |u|_\infty^{-\beta} dt \\
&\geq \frac{1}{2}||u||^2 12^{\beta/2} ||u||^{-\beta} = \frac{1}{2} 12^{\beta/2} ||u||^{2-\beta} \rightarrow +\infty, ||u|| \rightarrow 0.
\end{aligned}$$

For  $u_1 \in R^n, e(t)$  satisfying

$$\int_0^1 e(t)dt = 0, \int_0^1 |\dot{e}|^2 dt = 1, s > 0, |u_1|^2 + s^2 = R^2,$$

we claim for any  $t \in [0, 1]$ , we have

$$u_1 + se(t) \neq 0,$$

in fact, if otherwise, there is  $t_0$  such that  $e(t_0) = -u_1/s$  which is a contradiction since  $e(t) \in X_2$  and  $u_1 \in X_1$  and  $X_1$  is orthogonal to  $X_2, e(t) = 0$  for all  $t \in [0, 1]$ . So we have

$$\min_{t \in [0, 1]} |u_1 + se(t)| = m(R) > 0$$

We notice that

$$\begin{aligned}
&\max_{0 \leq t \leq 1} |u_1 + se(t)| \leq |u_1| + s \max |e(t)| \\
&\leq R + R 12^{-1/2} ||e|| = R(1 + 12^{-1/2}) = M(R)
\end{aligned}$$

Let

$$A_R \triangleq \sup \left\{ \int_0^1 -V(u_1 + se)dt, u_1 \in R^n, \int_0^1 e(t)dt = 0, \int_0^1 |\dot{e}|^2 dt = 1, s > 0, |u_1|^2 + s^2 = R^2 \right\}$$

Then

$$A_R \leq \max\{|V(x)|, m(R) \leq |x| \leq M(R)\} \triangleq B(R) < +\infty$$

$$\begin{aligned}
f(u_1 + se) &= \frac{1}{2} s^2 \int_0^1 (h - V(u_1 + se))dt \\
&= \frac{h}{2} s^2 + \frac{1}{2} s^2 \int_0^1 (-V(u_1 + se))dt \\
&\leq \frac{h}{2} R^2 + \frac{1}{2} R^2 B(R)
\end{aligned}$$

It's easy to know that there are  $\delta > 0$  small enough and  $R > \delta$  such that when  $||u|| = \delta$ , we have

$$f|_S \geq \frac{1}{2} 12^{\beta/2} \delta^{2-\beta} > \frac{h}{2} R^2 + \frac{1}{2} R^2 B(R)$$

In fact, for any given  $h > 0, R > 0$ , since  $\beta > 2$ , we can choose  $\delta > 0$  small enough such that  $\delta < R$  and

$$\frac{1}{2} 12^{\beta/2} \delta^{2-\beta} > \frac{h}{2} R^2 + \frac{1}{2} R^2 B(R)$$

It's easy to know  $f|_{\{u_1 \in R^n ||u_1| \leq R\}} = 0$ . So we have

$$\begin{aligned} f|_{\partial Q} &\leq \frac{h}{2}R^2 + \frac{1}{2}R^2B(R). \\ &< \frac{1}{2}12^{\beta/2}\delta^{2-\beta} \leq f|_S. \end{aligned}$$

## Remark

It's more interesting that we use the above ideas and methods and combine other methods to study the  $n(n \geq 2)$  body problems with weak force potentials.

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